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Thermosolutal convection in a nonlinear magnetic fluid

Abdullah A. Abdullah *

Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al-Qura University, P.O. Box 6337, Makkah, Saudi Arabia

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Abstract — This work examines the thermosolutal convection of a horizontal layer of an incompressible viscous magnetohydrodynamic fluid of variable permeability for the cases when the fluid is heated and soluted from above, heated and soluted from below, heated from below and soluted from above and heated from above and soluted from below. The model has been proposed by Roberts in the context of neutron stars but the results obtained are also relevant to the area of ferromagnetic fluids. The pressence of the variable magnetic permeability has no effect on the development of instabilities through the mechanism of stationary convection but influences the threshold of overstable convection which is often the preferred mechanism in nonterrestrial applications. In the context of ferromagnetic fluids, both stationary and overstable instability can be expected to be realisable possibilites. When the fluid is heated from above and soluted from below the only possible mode of instability is overstability. However, for the other three cases stationary convection and overstability are possible under some conditions. Numerical results were obtained using series expansion of Chebyshev polynomials. © 2000 Éditions scientifiques et médicales Elsevier SAS

thermosolutal convection / magnetic field / magnetohydrodynamic fluid / ferromagnetic fluid

Nomenclature

a	dimensionless wave number	
$\mathbf{b}$	perturbation in magnetic field	T
$\mathbf{B}$	magnetic induction	T
C	perturbation in solute concentration	
C_ρ	solute mass concentration	$\text{kg}\cdot\text{m}^{-2}$
$\mathbf{E}$	electric field	$\text{V}\cdot\text{m}^{-1}$
g	acceleration due to gravity	$\text{m}\cdot\text{s}^{-2}$
$\mathbf{H}$	magnetic field	$\text{A}\cdot\text{m}^{-1}$
H'	constant	
$\mathbf{J}$	current density	$\text{A}\cdot\text{m}^{-2}$
M'	constant	
P	modified pressure	Pa
P_r	mechanical Prandtl number	
P'_r	solute Prandtl number	
P_m	magnetic Prandtl number	
Q	Chandrasekhar number	
R_t	thermal Rayleigh number	
R_s	solute Rayleigh number	
S_0	solute concentration on $x_3 = 0$	$\text{kg}\cdot\text{m}^{-2}$
T	absolute temperature	K
T_0	absolute temperature on $x_3 = 0$	K
$\mathbf{u}$	fluid velocity	$\text{m}\cdot\text{s}^{-1}$
$\mathbf{v}$	perturbation in velocity	$\text{m}\cdot\text{s}^{-1}$

Greek symbols

α	coefficent of volume expansion	K^{-1}
α'	coefficient of solvent expansion	K^{-1}
β	adverse temperature gradient	$\text{m}\cdot\text{K}^{-1}$
β'	adverse concentration gradient	$\text{m}\cdot\text{K}^{-1}$
ε	dimensionless magnetic number	
ϕ	magnetic susceptibility	
κ	coefficient of thermal diffusivity	$\text{m}^2\cdot\text{s}^{-1}$
κ'	coefficient of solute diffusivity	$\text{m}^2\cdot\text{s}^{-1}$
μ	magnetic permeability	$\text{H}\cdot\text{m}^{-1}$
ν	kinematic viscosity	$\text{m}^2\cdot\text{s}^{-1}$
θ	perturbation in temperature	K
ξ	vorticity of fluid	s^{-1}

* abdullah@uqu.edu.sa

1. INTRODUCTION

Thermal instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many areas of fluid dynamics. The earliest experiments to demonstrate the onset of thermal instability in fluids are attributed to Benard [1, 2].

The theoretical foundations for a correct interpretation of the foregoing facts were laid by Rayleigh [3]. Jeffreys [4, 5], Low [6] and Pellew and Southwell [7] extended Rayleigh's work for different boundary conditions. Thermal instability theory has been enlarged by the interest in hydrodynamic flows of electrically conducting fluids in the presence of a magnetic field. The presence of such fields in an electrically conducting fluid usually has the effect of inhibiting the development of instabilities. Benard convection of a magnetohydrodynamic fluid in the presence of a linear magnetic field has been examined by Thomson [8], Chandrasekhar [9, 10], Nakagawa [11, 12] and others.

The problem of thermosolutal convection in a layer of fluid heated from above has been studied by Stern [13] when the solute is salt and the solute concentration is increasing upwards. Veronis [14] studied Stern's problem when the fluid layer is heated and soluted from below. The same problem has been considered by Nield [15] when the fluid layer is heated from below and soluted from above. Sharma and Sharma [16] investigated analytically the Benard convection in a layer of conducting fluid heated and soluted from below through porous medium in the presence of a uniform vertical magnetic field. Thermosolutal convection in a heterogeneous fluid layer in porous medium has been studied by Khare and Sahai [17]. They found that heterogeneity has destabilizing effect on growth rates. They also discussed the same problem in the presence of magnetic field [18].

All previous studies have discussed Benard convection for magnetohydrodynamic fluids which have a linear constitutive relationship between the magnetic field **H** and the magnetic induction **B**. However, nonlinear constitutive relation between **H** and **B** may be appropriate for certain classes of materials. Roberts [19] and Muzikar and Pethick [20] use a nonlinear magnetization law to model convection in a neutron star. Cowley and Rosensweig [21], Gailitis [22] and others use nonlinear magnetization laws to describe the properties of ferromagnetic fluids. Abdullah and Lindsay [23, 24] used Roberts' model to discuss the Benard convection in the presence of a vertical and nonvertical magnetic field. They showed that this nonlinear relationship has no effect on the development of instabilities through the mechanism of stationary convection but influences the onset of overstable convection.

The strength of the nonlinearity is measured by a nondimensional parameter ε where the classical case discussed by Chandrasekhar [25] corresponds to $\varepsilon = 0$ in the absence of the porosity. Appropriate values of ε can be obtained from Kaiser and Miskolczy [26], Chantrell et al. [27], Charles and Popplewell [28] and Popplewell et al. [29] and are listed in Abdullah and Lindsay [23]. Typical values range from 0 to 2 but they can be as high as 7.

In this paper we shall use Roberts' model to discuss analytically and numerically the thermosolutal convection in a horizontal layer of a conducting magnetohydrodynamic fluid for the cases when the fluid is heated and soluted from above, heated and soluted from below, heated from above and soluted from below and heated from below and soluted from above in the presence of a constant vertical magnetic field. The numerical computations are to be performed using series expansions in Chebyshev polynomials. This method was first introduced by Lanczos [30] and Clenshaw [31] and has been developed and extensively applied to ordinary differential equations by Fox [32], Fox and Parker [33], Orszag [34, 35], Orszag and Kells [36], Davis et al. [37, 38] and others. The method possesses excellent convergence characteristics and effectively exhibits exponential convergence rather than finite power convergence.

2. MATHEMATICAL FORMULATION

Consider a horizontal layer of an incompressible, thermally and electrically conducting viscous soluted fluid of density ρ. The fluid is subject to a constant gravitational acceleration in the negative x_3 direction. A constant magnetic field is imposed across the layer in the positive x_3 direction and x_1 and x_2 axes are selected from a right-handed system of cartesian coordinates in which the magnetic induction **B** has a representation such that $\mathbf{B} = (0, 0, B)$ (*figure 1*).

In order to fully describe the nature of this model we need to discuss the interaction between electromagnetic and mechanical effects and so we define **E** and **J** to be respectively the electric field and the current density. The most general, properly invariant, constitutive relationship between the magnetic field, the magnetic induction and the current density has the form

$$H_i = \rho\phi B_i$$

where $B = \sqrt{(B_i B_i)}$ is the magnitude of the magnetic induction and ϕ is related to the partial derivative of the magnetic free energy with respect to B (see Roberts [19]). The conventional magnetic permeability, μ, is given by

$$\mu(\rho, B) = (\rho\phi)^{-1}$$

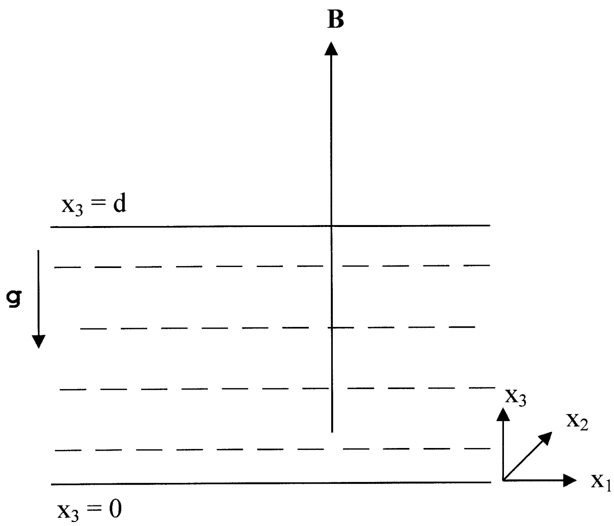

Figure 1. The coordinate system of the physical problem.

The strength of the nonlinear permeability is measured in terms of the nondimensional magnetic number ε where

$$\varepsilon = -\frac{B}{\mu(\rho, B)}\frac{\partial \mu}{\partial B}$$

Conventional ideas would indicate that the permeability is a decreasing function of B and so $\varepsilon \geq 0$. When $\varepsilon = 0$ then the magnetic permeability μ is constant and this area has been extensively researched by Chandrasekhar [25]. The magnetic variables are required to satisfy the Maxwell equations

$$\begin{aligned} \operatorname{div}\mathbf{B} &= \mathbf{B}_{i,i} = \mathbf{0} \\ \operatorname{curl}\mathbf{H} &= \mathbf{e}_{ijk}\mathbf{H}_{k,j} = \mathbf{J}_i \\ \operatorname{curl}\mathbf{E} &= \mathbf{e}_{ijk}\mathbf{E}_{k,j} = -\frac{\partial \mathbf{B}_i}{\partial t} \end{aligned} \tag{2.1}$$

where the displacement current has been neglected as is customary in this type of problems.

If we now use the Boussinesq approximation, namely that density is constant everywhere except in the body force term where the density is linearly proportional to temperature and solute concentration, i.e.

$$\rho = \rho_0\left(1 - \alpha T + \alpha' C_{\mathrm{e}}\right)$$

where T is the absolute temperature, α, α' are the coefficients of volume and solvent expansion and C_{e} is the solute mass concentration, then we may show that the relevant mechanical and thermal field equations take the form

$$\begin{aligned} u_{i,i} &= 0 \\ \frac{\mathrm{D}T}{\mathrm{D}t} &= \kappa \Delta T \\ \frac{\mathrm{D}C_{\mathrm{e}}}{\mathrm{D}t} &= \kappa' \Delta C_{\mathrm{e}} \\ \frac{\mathrm{D}u_i}{\mathrm{D}t} &= -(P/\rho)_{,i} + B_k(\phi B_i)_{,k} + \nu \Delta u_i \\ &\quad - g\left[1 - \alpha T + \alpha' C_{\mathrm{e}}\right]\delta_{i3} \end{aligned} \tag{2.2}$$

where u_i is the fluid velocity, ν is the kinematic viscosity, g is the acceleration due to gravity, $\mathrm{D}(\cdot)/\mathrm{D}t$ is the convected derivative, κ, κ' are the coefficients of thermal and solute diffusivity and

$$P = p + \frac{\rho}{2}\left(B^2\phi + \int B^2\phi_B \,\mathrm{d}B\right)$$

is the modified pressure. We may observe that equations (2.1) and (2.2) have a steady state solution in which

$$\begin{aligned} &\mathbf{u} = 0, \qquad \mathbf{E} = 0, \qquad \mathbf{J} = 0 \\ &T = T_0 - \beta x_3, \qquad C_{\mathrm{e}} = S_0 - \beta' x_3 \\ &P = P(x_3), \qquad \mathbf{B} = (0, 0, B), \qquad B = \text{const} \end{aligned} \tag{2.3}$$

where T_0, S_0 are the temperature and solute concentration on the plane $x_3 = 0$ and β, β' are the adverse temperature and concentration gradients. Let **v**, **J**, θ, C, P and **b** be respectively perturbations in velocity, current density, temperature, solute concentration, pressure and magnetic induction about their equilibrium values described in (2.3), then we may verify that the linearized version of equations (2.1) and (2.2) are

$$\begin{aligned} \frac{\partial v_i}{\partial t} &= -(P/\rho)_{,i} + B\phi b_{i,3} + B^2\phi_B b_{3,3}\delta_{i3} \\ &\quad + \nu \Delta v_i + g(\alpha\theta - \alpha' C)\delta_{i3} \\ b_{i,i} &= 0 \\ v_{i,i} &= 0 \\ \frac{\partial \theta}{\partial t} - \beta v_3 &= \kappa \Delta \theta \\ \frac{\partial C}{\partial t} - \beta' v_3 &= \kappa' \Delta C \\ J_i &= e_{ijk}(\rho\phi b_k + \rho B\phi_B b_3\delta_{k3})_{,j} \\ \frac{\partial b_i}{\partial t} &= B v_{i,3} - \eta e_{ijk} J_{k,j} \end{aligned} \tag{2.4}$$

At this stage we introduce dimensionless variables $\hat{x}_i$, $\hat{v}_i$, $\hat{t}$, $\hat{\theta}$, $\widehat{C}$, $\widehat{P}$, $\hat{b}_i$ and $\widehat{J}_i$ where

$$x_i = d\hat{x}_i, \qquad v_i = \frac{\kappa}{d}\hat{v}_i$$
$$t = \frac{d^2}{\nu}\hat{t}, \qquad \theta = \frac{\kappa}{d}\sqrt{\frac{\nu|\beta|}{\kappa\alpha g}}\,\hat{\theta}$$
$$C = \frac{\kappa}{d}\sqrt{\frac{\nu|\beta'|}{\kappa'\alpha' g}}\,\widehat{C}, \qquad P = \frac{\kappa\nu\rho}{d^2}\widehat{P}$$
$$b_i = \frac{\kappa\nu}{B\phi d^2}\hat{b}_i, \qquad J_i = \frac{\kappa\nu\rho}{Bd^3}\widehat{J}_i$$

After this nondimensionalization, equations (2.4) are simplified to

$$\begin{aligned}
\frac{\partial v_i}{\partial t} &= -P_{,i} + b_{i,3} + \varepsilon b_{3,3}\delta_{i3} \\
&\quad + \Delta v_i + \sqrt{R_\mathrm{t}}\,\theta\delta_{i3} - \sqrt{R_\mathrm{s}}\,C\delta_{i3} \\
b_{i,i} &= 0 \\
v_{i,i} &= 0 \\
P_\mathrm{r}\frac{\partial\theta}{\partial t} &+ H'\sqrt{R_\mathrm{t}}v_3 = \Delta\theta \\
P'_\mathrm{r}\frac{\partial C}{\partial t} &+ M'\sqrt{R_\mathrm{s}}v_3 = \Delta C \\
J_i &= e_{ijk}(b_{k,j} + \varepsilon b_{3,3}\delta_{k3}) \\
P_\mathrm{m}\frac{\partial b_i}{\partial t} &= Qv_{i,3} - e_{ijk}J_{k,j}
\end{aligned} \tag{2.5}$$

where the $\hat{}$ superscript has been dropped but all the variables are now nondimensional and where the nondimensional numbers R_t, R_s, ε, P_r, P'_r, P_m and Q are given by

$$R_\mathrm{t} = \frac{g\alpha|\beta|}{\kappa\nu}d^4, \qquad R_\mathrm{s} = \frac{g\alpha'|\beta'|}{\kappa'\nu}d^4, \qquad \varepsilon = \frac{B\phi_B}{\phi}$$
$$P_\mathrm{r} = \frac{\nu}{\kappa}, \qquad P'_\mathrm{r} = \frac{\nu}{\kappa'}, \qquad P_\mathrm{m} = \frac{\nu}{\rho\eta\phi}$$
$$Q = \frac{B^2d^2}{\rho\eta\nu}$$

and where

$$H' = -\frac{\beta}{|\beta|} = \begin{cases} +1 & \text{when heating from above} \\ -1 & \text{when heating from below} \end{cases}$$

$$M' = -\frac{\beta'}{|\beta'|} = \begin{cases} +1 & \text{when solute concentration increases upwards} \\ -1 & \text{when solute concentration decreases upwards} \end{cases}$$

3. THE BOUNDARY CONDITIONS

The fluid is confined between the planes $x_3 = 0$ and $x_3 = 1$ and on these planes we need to specify mechanical, thermal, solute and electromagnetic conditions.

3.1. Mechanical conditions

Regardless of the nature of these boundaries we require

$$u_3 = 0 \quad \text{on } x_3 = 0, 1$$

i.e. the normal component of velocity must vanish on these surfaces. If we define ξ to be the fluid vorticity then $\xi = \mathrm{curl}\,\mathbf{u}$ and the mechanical boundary conditions can be expressed by the equations

$$\xi_3 = 0 \quad \text{on a rigid boundary}$$
$$\frac{\partial\xi_3}{\partial x_3} = 0 \quad \text{on a free boundary}$$

If the fluid is also incompressible, then u_3 satisfies the additional property

$$\frac{\partial u_3}{\partial x_3} = 0 \quad \text{on a rigid boundary}$$
$$\frac{\partial^2 u_3}{\partial x_3^2} = 0 \quad \text{on a free boundary}$$

3.2. Thermal conditions

At a perfectly conducting boundary, the temperatures of the boundary and impinging fluid match whereas on a perfectly insulating boundary, no heat transfer can take place between the fluid and the surroundings and thus the normal derivative of temperature is zero. In mathematical terms, the possible thermal conditions are

$$T = T_\mathrm{ext} \quad \text{on a conducting boundary}$$
$$\frac{\partial T}{\partial x_3} = 0 \quad \text{on an insulating boundary}$$
$$\frac{\partial T}{\partial x_3} + h(T - T_\mathrm{ext}) = 0 \quad \text{on a radiating boundary}$$

where h is a heat transfer coefficient, sometimes known as the Robin constant, and T_ext is the temperature of the region exterior to the fluid boundary.

3.3. Solute conditions

These conditions can be expressed in mathematical terms as

$$
\begin{aligned}
C_{\mathrm{e}} &= C_{\mathrm{ext}} \quad \text{on a permeable boundary}\\
C_{\mathrm{e},3} &= 0 \quad \text{on an impermeable boundary}
\end{aligned}
$$

where C_{ext} is the solute concentration in the region exterior to the fluid boundary.

3.4. Electromagnetic conditions

On a perfectly insulating electromagnetic boundary, no current can flow to the exterior region and the magnetic field is continuous across the boundary with the external magnetic field being derived from a scalar potential since curl $\mathbf{H} = 0$ in the exterior region. On a stationary perfectly conducting boundary, the surface components of electric field are zero as is the time derivative of the normal component of the magnetic induction. It is common practice to associate mechanically rigid and electrically perfectly conducting stationary boundaries. In this case the surface components of current density are also zero and so the electromagnetic boundary conditions assume the form

$$
\frac{\partial b_3}{\partial t} = 0, \qquad \frac{\partial J_3}{\partial x_3} = 0
$$

4. THE EIGENVALUE PROBLEM

We aim to investigate the linear stability of the conduction solution (2.3) and with this aim in mind we construct the related eigenvalue problem from equations (2.5) and the boundary conditions. As is the case in many convection problems, vector components parallel to the direction of gravity play a central role and so it is convenient to introduce the variables w, b, J and ξ by the definitions

$$
\begin{aligned}
&w = v_3, \qquad b = b_3, \qquad J = J_3\\
&\xi = \xi_3, \qquad z = x_3
\end{aligned}
\tag{4.1}
$$

We now look for a solution of the form

$$
\psi = \psi(z)\exp\big[\mathrm{i}(nx + my) + \sigma t\big]
$$

Thus, the relative equations become

$$
\begin{aligned}
\sigma L w &= L^2 w - a^2\sqrt{R_{\mathrm{t}}}\,\theta + a^2\sqrt{R_{\mathrm{s}}}\,C\\
&\quad + L(\mathrm{D}b) - \varepsilon a^2(\mathrm{D}b)\\
\sigma P_{\mathrm{m}} b &= Q\,\mathrm{D}w + Lb - \varepsilon a^2 b\\
\sigma P_{\mathrm{r}}\theta &= L\theta - H'\sqrt{R_{\mathrm{t}}}\,w\\
\sigma P_{\mathrm{r}}' C &= LC - M'\sqrt{R_{\mathrm{s}}}\,w
\end{aligned}
\tag{4.2}
$$

where D is the operator $\mathrm{d}/\mathrm{d}z$, a^2 $(= n^2 + m^2)$ is the wave number and L is the operator $(\mathrm{D}^2 - a^2)$. We may eliminate C, θ and b from (4.2) to obtain

$$
\begin{aligned}
&\sigma^4 P_{\mathrm{m}} P_{\mathrm{r}} P_{\mathrm{r}}' L w - \sigma^3\big[\big(P_{\mathrm{m}} P_{\mathrm{r}} P_{\mathrm{r}}' + P_{\mathrm{m}} P_{\mathrm{r}}\\
&\quad + P_{\mathrm{m}} P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}'\big) L^2 w - \varepsilon a^2 P_{\mathrm{r}} P_{\mathrm{r}}' L w\big]\\
&\quad + \sigma^2\big\{\big(P_{\mathrm{r}} + P_{\mathrm{r}} P_{\mathrm{m}} + P_{\mathrm{m}} + P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}' + P_{\mathrm{m}} P_{\mathrm{r}}'\big) L^3 w\\
&\quad - \big[Q P_{\mathrm{r}} P_{\mathrm{r}}' + \varepsilon a^2\big(P_{\mathrm{r}} + P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}'\big)\big] L^2 w\\
&\quad + Q a^2(\varepsilon - 1) P_{\mathrm{r}} P_{\mathrm{r}}' L w + \big[a^2 P_{\mathrm{m}}\big(M' P_{\mathrm{r}} R_{\mathrm{s}} - P_{\mathrm{r}}' R_{\mathrm{t}} H'\big)\\
&\quad + Q\varepsilon a^4 P_{\mathrm{r}} P_{\mathrm{r}}'\big] w\big\} + \sigma\big\{-\big(1 + P_{\mathrm{r}} + P_{\mathrm{r}}' + P_{\mathrm{m}}\big) L^4 w\\
&\quad + \big[\varepsilon a^2\big(1 + P_{\mathrm{r}} + P_{\mathrm{r}}'\big) + Q\big(P_{\mathrm{r}} + P_{\mathrm{r}}'\big)\big] L^3 w\\
&\quad - Q a^2(\varepsilon - 1)\big(P_{\mathrm{r}} + P_{\mathrm{r}}'\big) L^2 w + \big[H' R_{\mathrm{t}} a^2\big(P_{\mathrm{r}}' + P_{\mathrm{m}}\big)\\
&\quad - M' a^2(P_{\mathrm{r}} + P_{\mathrm{m}}) - Q\varepsilon a^4\big(P_{\mathrm{r}} + P_{\mathrm{r}}'\big)\big] L w\\
&\quad + \varepsilon a^4\big(M' R_{\mathrm{s}} P_{\mathrm{r}} - H' R_{\mathrm{t}} P_{\mathrm{r}}'\big) w\big\} + L^5 w\\
&\quad - \big(Q + \varepsilon a^2\big) L^4 w + Q a^2(\varepsilon - 1) L^3 w\\
&\quad + a^2\big(M' R_{\mathrm{s}} - H' R_{\mathrm{t}} + Q\varepsilon a^2\big) L^2 w\\
&\quad + \varepsilon a^4\big(H' R_{\mathrm{t}} - M' R_{\mathrm{s}}\big) L w = 0
\end{aligned}
\tag{4.3}
$$

which is a tenth order ordinary differential equation to be satisfied by w. In the following analysis we shall consider both boundaries to be free but later on we present results for the corresponding rigid boundary value problems. For the free boundary value problems

$$
w = \mathrm{D}^2 w = 0 \quad \text{on } z = 0, 1
$$

thus equation (4.3) has eigenfunctions $w = A\sin(l\pi z)$. Consequently, $Lw = -\lambda w$ where $\lambda = l^2\pi^2 + a^2$ and σ satisfies the quartic equation

$$
\begin{aligned}
&\sigma^4 P_{\mathrm{m}} P_{\mathrm{r}} P_{\mathrm{r}}' - \sigma^3\big[\big(P_{\mathrm{m}} P_{\mathrm{r}} P_{\mathrm{r}}' + P_{\mathrm{m}} P_{\mathrm{r}}\\
&\quad + P_{\mathrm{m}} P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}'\big)\lambda + \varepsilon a^2 P_{\mathrm{r}} P_{\mathrm{r}}'\big]\\
&\quad + \sigma^2\big[\big(P_{\mathrm{r}} + P_{\mathrm{r}} P_{\mathrm{m}} + P_{\mathrm{m}} + P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}' + P_{\mathrm{m}} P_{\mathrm{r}}'\big)\lambda^2\\
&\quad + Q P_{\mathrm{r}} P_{\mathrm{r}}' l^2\pi^2\big(\lambda + \varepsilon a^2\big)\lambda^{-1} + \varepsilon a^2\big(P_{\mathrm{r}} + P_{\mathrm{r}}' + P_{\mathrm{r}} P_{\mathrm{r}}'\big)\lambda\\
&\quad + a^2 P_{\mathrm{m}}\big(H' R_{\mathrm{t}} P_{\mathrm{r}}' - M' R_{\mathrm{s}} P_{\mathrm{r}}\big)\lambda^{-1}\big]\\
&\quad + \sigma\big\{\big(1 + P_{\mathrm{r}} + P_{\mathrm{r}}' + P_{\mathrm{m}}\big)\lambda^3 + \varepsilon a^2\big(1 + P_{\mathrm{r}} + P_{\mathrm{r}}'\big)\lambda^2\\
&\quad + Q l^2\pi^2\big(P_{\mathrm{r}} + P_{\mathrm{r}}'\big)\big(\lambda + \varepsilon a^2\big)
\end{aligned}
$$

$$+a^2\big[H'R_t\big(P_m+P'_r\big)-M'R_s(P_m+P_r)\big]$$
$$+\varepsilon a^4\big(H'R_tP'_r-M'R_sP_r\big)\lambda^{-1}\big\}+\lambda^4+\varepsilon a^2\lambda^3$$
$$+Ql^2\pi^2\lambda\big(\lambda+\varepsilon a^2\big)+a^2\big(H'R_t-M'R_s\big)\lambda$$
$$+\varepsilon a^4\big(H'R_t-M'R_s\big)=0 \tag{4.4}$$

The solutions of (4.4) are functions of P_r, P'_r, P_m, ε, Q, R_t and R_s and we have to examine how the nature of these solutions depends on those parameters by discussing the following cases.

Case 1. Fluid is heated from above and soluted from below

Here $H'=1$ and $M'=-1$, and we need to discuss the roots of the polynomial equation $f(\sigma)=0$ where the form of $f(\sigma)$ is obtained from equation (4.4). Let us assume that $|\mathbf{B}|$ is an increasing function of $|\mathbf{H}|$ then $\mathrm{d}(\phi B)/\mathrm{d}B>0$ from which we can show that $1+\varepsilon>0$. Hence $\lambda+\varepsilon a^2>0$. Clearly, all the coefficients of $f(\sigma)$ are positive and real. Thus $f(\sigma)=0$ has either four negative real solutions (i.e. no instability happens), two negative real solutions and two complex conjugate pair solutions or four complex conjugate pair solutions. In the second case we have overstability since the real part of the complex conjugate pair solutions is positive and this can be explained in the following way. Let Σ be the sum of the roots of $f(\sigma)=0$ then

$$\Sigma=-\frac{\lambda}{P_rP_mP'_r}\big(P_mP_rP'_r+P_rP_m+P'_rP_m+P_rP'_r\big)$$

and we can show that $f(\Sigma)>0$. Let σ_1, σ_2 be the two real roots and let σ_3 be the real part of the complex conjugate pair roots of $f(\sigma)$ then $\Sigma=\sigma_1+\sigma_2+2\sigma_3$ and since $f'(\sigma)>0$ then $\Sigma-\sigma_1>0$ and so $\sigma_2+2\sigma_3>0$ hence $\sigma_3>0$. In the third case we may have overstability and this can be explained by assuming that σ_1, σ_2 are the real parts of the complex roots then $\Sigma=\sigma_1+\sigma_2$ and so $\sigma_1+\sigma_2<0$, hence, either both are negative (i.e. no overstability) or one of them is positive (i.e. overstability happens). Since it is quite difficult to discuss the roots of the polynomial equation $f(\sigma)=0$ for the following cases then we shall restrict the analysis to produce the thermal Rayleigh number for the cases of stationary convection and overstability.

Case 2. Fluid is heated and soluted from above

In this case $H'=1$ and $M'=1$ and the thermal Rayleigh number for the onset of stationary convection can be obtained from equation (4.4) when $\sigma=0$. Thus

$$R_t=-\frac{\lambda}{a^2}\big(\lambda^2+Q\pi^2\big)+R_s$$

It is clear that stationary convection is possible provided

$$R_s>\frac{\lambda}{a^2}\big(\lambda^2+Q\pi^2\big)$$

To obtain the thermal Rayleigh number for the overstability case we put

$$\sigma=\mathrm{i}\sigma_1\pi^2,\qquad a^2=a_1\pi^2,\qquad Q=Q_1\pi^2$$
$$R_t=R_{t1}\pi^4,\qquad R_s=R_{s1}\pi^4,\qquad l^2=1$$

where σ is complex and $\sigma\neq 0$. Substituting into (4.4) we obtain

$$R_{t1}=-\frac{1+a_1}{a_1}(1+a_1+\mathrm{i}\sigma_1)(1+a_1+\mathrm{i}\sigma_1P_r)$$
$$+R_{s1}\frac{1+a_1+\mathrm{i}\sigma_1P_r}{1+a_1+\mathrm{i}\sigma_1P'_r}$$
$$-\frac{Q_1(1+a_1+\mathrm{i}\sigma_1P_r)[1+a_1(1+\varepsilon)]}{a_1[1+\mathrm{i}\sigma_1P_m+a_1(1+\varepsilon)]} \tag{4.5}$$

By equating separately the real and imaginary parts of equation (4.5) and eliminating R_{t1} from the resulting equations we obtain a quadratic equation in Y $(=\sigma_1^2)$ such that

$$A_2Y^2+A_1Y+A_0=0$$

where

$$A_2=P_m^2P_r'^2(1+P_r)(1+a_1)^2$$
$$A_1=P_m^2(1+P_r)(1+a_1)^4$$
$$+P_r'^2(1+P_r)(1+a_1)^2\big[1+a_1(1+\varepsilon)\big]^2$$
$$+R_{s1}P_m^2a_1(1+a_1)\big(P_r-P'_r\big)$$
$$+Q_1P_r'^2\big[1+a_1(1+\varepsilon)\big]$$
$$\cdot\big\{P_r\big[1+a_1(1+\varepsilon)\big]-P_m(1+a_1)\big\}$$
$$A_0=(1+P_r)(1+a_1)^4\big[1+a_1(1+\varepsilon)\big]^2$$
$$+a_1R_{s1}(1+a_1)\big(P_r-P'_r\big)\big[1+a_1(1+\varepsilon)\big]^2$$
$$+Q_1(1+a_1)^2\big[1+a_1(1+\varepsilon)\big]$$
$$\cdot\big\{P_r\big[1+a_1(1+\varepsilon)\big]-P_m(1+a_1)\big\}$$

A necessary condition for overstability to happen is that $A_0<0$ and this can be satisfied if either case of the following is satisfied:

(i)

$$P_m > P_r\left[1+\varepsilon\frac{a_1}{1+a_1}\right], \qquad P_r' > P_r$$

$$Q_1 > \frac{[1+a_1(1+\varepsilon)][(1+P_r)(1+a_1)^3 - R_{s1}a_1(P_r - P_r')]}{(1+a_1)\{P_r[1+a_1(1+\varepsilon)] - P_m(1+a_1)\}}$$

(ii)

$$P_m > P_r\left[1+\varepsilon\frac{a_1}{1+a_1}\right], \qquad P_r' = P_r$$
$$Q_1 > \frac{(1+P_r)(1+a_1)^2[1+a_1(1+\varepsilon)]}{\{P_r[1+a_1(1+\varepsilon)] - P_m(1+a_1)\}} \tag{4.6}$$

(iii)

$$P_m > P_r\left[1+\varepsilon\frac{a_1}{1+a_1}\right], \qquad P_r' < P_r$$

$$Q_1 > \frac{[1+a_1(1+\varepsilon)][(1+P_r)(1+a_1)^3 + R_{s1}a_1(P_r - P_r')]}{(1+a_1)\{P_r[1+a_1(1+\varepsilon)] - P_m(1+a_1)\}}$$

To study the effects of solute concentration and magnetic field on the thermal Rayleigh number we can see from (4.5) that

$$\frac{dR_{t1}}{dR_{s1}} = 1, \qquad \frac{dR_{t1}}{dQ_1} = -\frac{1+a_1}{a_1}$$
$$\frac{dR_{t1}}{d\varepsilon} = -\frac{Q_1(1+a_1)}{2[1+a_1(1+\varepsilon)]}$$

i.e. for overstability, the solute concentration has a stabilizing effect while the magnetic field has a destabilizing effect.

Case 3. Fluid is heated from below and soluted from above

In this case $H' = -1$ and $M' = 1$ and the thermal Rayleigh number for the onset of stationary convection can be obtained from equation (4.4) when $\sigma = 0$. Thus

$$R_t = \frac{\lambda}{a^2}\left(\lambda^2 + Q\pi^2\right) - R_s$$

It is clear that stationary convection is possible provided

$$R_s < \frac{\lambda}{a^2}\left(\lambda^2 + Q\pi^2\right)$$

Following the same analysis as in the previous case we can see that overstability occurs if any case of (4.6) is satisfied. Moreover, we can see from (4.5) that

$$\frac{dR_{t1}}{dR_{s1}} = -1, \qquad \frac{dR_{t1}}{dQ_1} = \frac{1+a_1}{a_1}$$
$$\frac{dR_{t1}}{d\varepsilon} = \frac{Q_1(1+a_1)}{2[1+a_1(1+\varepsilon)]}$$

i.e. for ovestability, the solute concentration has a destabilizing effect while the magnetic effect has a stabilizing effect. It appears also that the nonlinearity has a profound effect on the stability of the system.

Case 4. Fluid is heated and soluted from below

In this case $H' = -1$ and $M' = -1$ and the thermal Rayleigh number for the onset of stationary convection can be obtained from equation (4.4) when $\sigma = 0$. Thus

$$R_t = \frac{\lambda}{a^2}\left(\lambda^2 + Q\pi^2\right) + R_s$$

Following the same analysis as in the previous case we can see that overstability occurs if any case of (4.6) is satisfied. Moreover, we can see from (4.5) that

$$\frac{dR_{t1}}{dR_{s1}} = 1, \qquad \frac{dR_{t1}}{dQ_1} = \frac{1+a_1}{a_1},$$
$$\frac{dR_{t1}}{d\varepsilon} = \frac{Q_1(1+a_1)}{2[1+a_1(1+\varepsilon)]}.$$

i.e. for ovestability, the solute concentration and the magnetic effect have a stabilizing effect. It also appears that the nonlinearity has a profound effect on the stability of the system in this case.

5. RESULTS AND DISCUSSION

In the previous section, several cases have been discussed in the context of heating and solute concentrating of the fluid from above or below. It has been shown that for all cases, the thermal and solute Rayleigh numbers are independent of the strength of the nonlinearity, ε, for stationary convection case. However, this nonlinearity has a profound effect on the development of instabilities through overstable mode for all cases. When the fluid is heated from above and soluted from below no instability ensues through stationary convection but we may have overstability under some conditions. When the fluid is heated and soluted from above we could have both stationary convection and overstability, and in this case the solute concentration has a stabilizing effect but the magnetic field has a destabilizing effect. When the fluid is heated from below and soluted from above we could have both stationary convection and overstability, and in this case the magnetic field has a stabilizing effect but the solute concentration has a destabilizing effect on the system. Finally, when the fluid is heated and soluted from below both stationary convection and overstability are possible and both have a stabilizing effect on the system.

Equations (4.2) and the boundary conditions constitute a tength-order eigenvalue problem. This system is solved numerically using expansion in Chebyshev polynomials when the fluid is heated from below and soluted from above. The critical thermal and solute Rayleigh numbers are obtained by minimizing over the wave number for various assigned values of the parameters Q, P_r, P'_r, P_m, and ε. According to the boundary conditions of the solute concentration we have two cases to consider:

$$\text{case (a):}\quad \mathrm{D}C = 0 \quad \text{at } x_3 = 0,\ x_3 = d$$

$$\text{case (b):}\quad \mathrm{D}C = 0 \quad \text{at } x_3 = 0$$

$$C = 0 \quad \text{at } x_3 = d$$

The relation between the thermal and solute Rayleigh numbers, R_t, R_s, for the overstable convection is displayed in *figures 2* and *3* for case (a) and case (b) of solute concentration boundary conditions, respectively, when both boundaries are rigid for various values of the parameter ε. It is clear that the nonlinear relation between the magnetic field and the magnetic induction has a profound effect where the case of linear relation corresponds to the graph of $\varepsilon = 0$. It also appears from the figures that there is a converse relation between the thermal and solute Rayleigh numbers. The numerical results corresponding to *figures 2* and *3* are illustrated in *tables I* and *II*, respectively. The relation between Chandrasekhar number Q and the critical solute Rayleigh number is displayed in *figures 4* and *5* for case (a) and case (b) of solute

TABLE I
The relation between the thermal and solute Rayleigh numbers for the overstable case for case (a) of solute boundary conditions when both boundaries are rigid. Here $Q = 3000$, $P_\mathrm{m} = 4$, $P_\mathrm{r} = 1$, $P'_\mathrm{r} = 1$.

$\varepsilon = 1$		$\varepsilon = 0.5$		$\varepsilon = 0$		
R_s	a	R_s	a	R_s	a	R_t
–	–	29 617.349	6.648	22 956.277	6.675	0
–	–	29 079.884	6.642	22 430.012	6.670	500
–	–	28 542.424	6.635	21 903.724	6.665	1 000
36 107.303	6.781	28 004.970	6.629	21 377.414	6.660	1 500
35 545.445	6.771	27 467.525	6.623	20 851.082	6.654	2 000
34 983.566	6.760	26 930.089	6.616	20 324.730	6.649	2 500
34 421.676	6.749	26 392.666	6.610	19 798.357	6.644	3 000
33 859.786	6.738	25 855.256	6.604	19 271.965	6.639	3 500
33 297.605	6.728	25 317.862	6.598	18 745.555	6.634	4 000
32 736.044	6.717	24 780.486	6.591	18 219.128	6.628	4 500
32 174.212	6.706	24 243.129	6.585	17 692.684	6.623	5 000
31 612.421	6.695	23 705.794	6.579	17 166.225	6.618	5 500
31 050.680	6.685	23 168.481	6.573	16 639.750	6.613	6 000
30 488.998	6.674	22 631.193	6.567	16 113.262	6.608	6 500
29 927.387	6.663	22 093.932	6.561	15 586.760	6.603	7 000
29 365.855	6.653	21 556.699	6.555	15 060.252	6.600	7 500
28 804.413	6.642	21 019.496	6.549	14 533.720	6.593	8 000
28 243.069	6.632	20 482.435	6.543	14 007.184	6.588	8 500
27 702.914	6.621	19 945.187	6.537	13 480.638	6.583	9 000
27 120.717	6.611	19 408.083	6.531	12 954.083	6.578	9 500
26 559.727	6.601	18 871.016	6.525	12 427.520	6.573	10 000
25 998.872	6.590	18 333.863	6.519	11 900.949	6.568	10 500
25 438.161	6.580	17 796.998	6.514	11 374.372	6.563	11 000
24 877.603	6.570	17 260.050	6.508	10 847.792	6.556	11 500
24 317.205	6.560	16 723.144	6.502	10 321.200	6.553	12 000

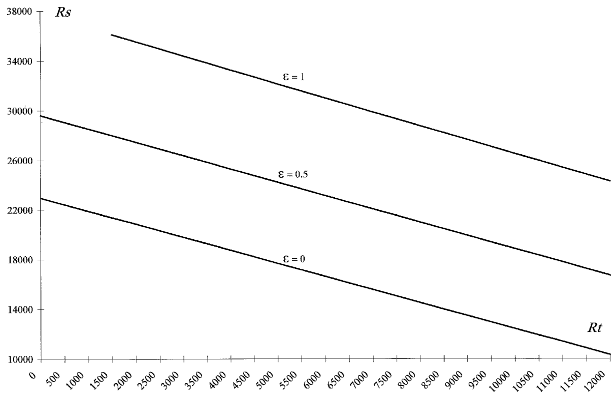

Figure 2. The relation between thermal and solute Rayleigh numbers for the overstable case for case (a) of solute boundary conditions when both boundaries are rigid. Here $Q = 3\,000$.

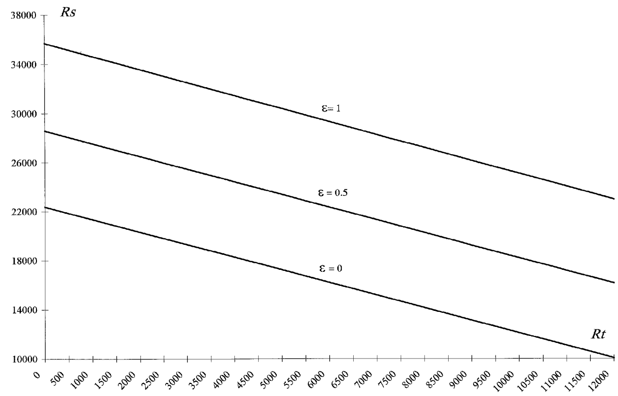

Figure 3. The relation between thermal and solute Rayleigh numbers for the overstable case for case (b) of solute boundary conditions when both boundaries are rigid. Here $Q = 3\,000$.

TABLE II
The relation between the thermal and solute Rayleigh numbers for the overstable case for case (b) of solute boundary conditions when both boundaries are rigid. Here $Q = 3\,000,\ P_{\mathrm{m}} = 4,\ P_{\mathrm{r}} = 1,\ P_{\mathrm{r}}' = 1$.

$\varepsilon = 1$		$\varepsilon = 0.5$		$\varepsilon = 0$		
R_{s}	a	R_{s}	a	R_{s}	a	R_{t}
35 697.418	6.460	28 578.825	6.479	22 375.157	6.562	0
35 167.780	6.456	28 060.272	6.476	21 861.735	6.560	500
34 638.241	6.451	27 541.750	6.473	21 348.323	6.557	1 000
34 108.802	6.447	27 023.258	6.471	20 834.922	6.555	1 500
33 579.462	6.443	26 504.796	6.468	20 321.532	6.552	2 000
33 050.221	6.438	25 986.365	6.465	19 808.152	6.550	2 500
32 521.080	6.434	25 467.965	6.463	19 294.783	6.547	3 000
31 992.037	6.430	24 949.595	6.460	18 781.425	6.545	3 500
31 463.094	6.425	24 431.255	6.457	18 268.078	6.543	4 000
30 934.249	6.421	23 912.946	6.455	17 754.742	6.540	4 500
30 405.503	6.417	23 394.667	6.452	17 241.416	6.538	5 000
29 876.855	6.413	22 876.419	6.449	16 728.102	6.535	5 500
29 348.306	6.409	22 358.201	6.447	16 214.798	6.533	6 000
28 819.855	6.405	21 840.013	6.444	15 701.506	6.531	6 500
28 291.502	6.401	21 321.856	6.441	15 188.224	6.528	7 000
27 763.246	6.397	20 803.729	6.439	14 674.953	6.526	7 500
27 235.088	6.393	20 285.632	6.436	14 161.694	6.524	8 000
26 707.027	6.389	19 767.566	6.434	13 648.445	6.521	8 500
26 179.062	6.385	19 249.529	6.431	13 135.208	6.519	9 000
25 651.195	6.381	18 731.523	6.429	12 621.982	6.517	9 500
25 123.423	6.377	18 213.547	6.426	12 108.766	6.514	10 000
24 595.748	6.373	17 695.601	6.424	11 595.563	6.512	10 500
24 068.168	6.370	17 177.685	6.421	11 082.370	6.510	11 000
23 540.683	6.366	16 659.800	6.419	10 569.188	6.507	11 500
23 013.293	6.362	16 141.944	6.416	10 056.018	6.505	12 000

concentration boundary conditions when both boundaries are rigid. It is clear from the figures that the magnetic field has a stabilizing effect on this system and that the strength of the nonlinearity, ε, has a profound effect where the linear case between the magnetic field and the magnetic induction corresponds to the graph of $\varepsilon = 0$. The numerical

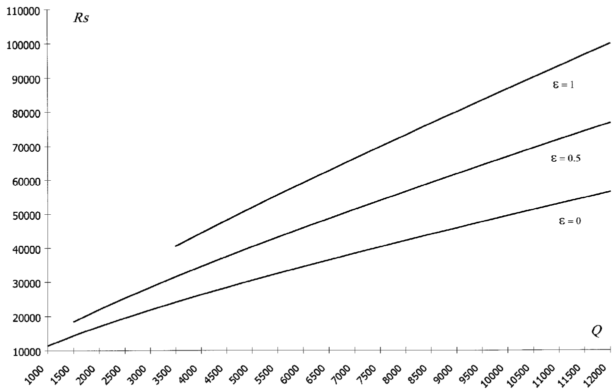

Figure 4. The relation between Chandrasekhar number and solute Rayleigh numbers for the overstable case for case (a) of solute boundary conditions when both boundaries are rigid. Here $R_t = 1\,000$.

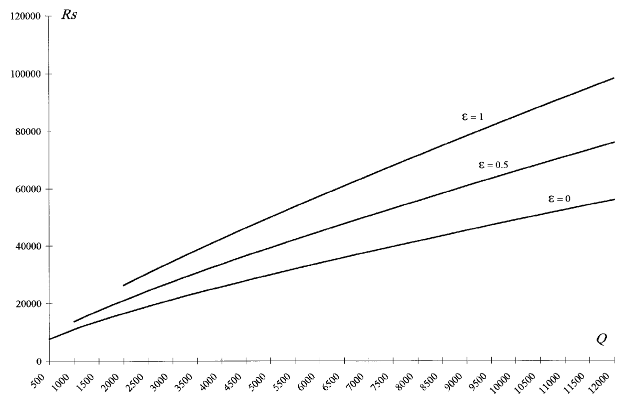

Figure 5. The relation between Chandrasekhar number and solute Rayleigh numbers for the overstable case for case (b) of solute boundary conditions when both boundaries are rigid. Here $R_t = 1\,000$.

TABLE III
The relation between Chandrasekhar number and solute Rayleigh number for the overstable case for case (a) of solute boundary conditions when both boundaries are rigid. Here $R_t = 1\,000$, $P_m = 4$, $P_r = 1$, $P_r' = 1$.

$\varepsilon = 1$		$\varepsilon = 0.5$		$\varepsilon = 0$		
R_s	a	R_s	a	R_s	a	Q
–	–	–	–	11 458.310	5.678	1 000
–	–	18 442.917	6.044	14 400.818	6.020	1 500
–	–	22 004.652	6.280	17 062.524	6.277	2 000
–	–	25 349.126	6.471	19 546.102	6.486	2 500
–	–	28 542.424	6.653	21 903.724	6.665	3 000
40 634.219	6.902	31 622.796	6.780	24 166.142	6.821	3 500
44 495.356	7.003	34 614.555	6.911	26 353.204	6.961	4 000
48 274.023	7.097	37 535.237	7.031	28 478.602	7.089	4 500
51 984.974	7.187	40 393.681	7.142	30 552.250	7.206	5 000
55 638.802	7.273	43 201.708	7.245	32 581.605	7.314	5 500
59 243.405	7.355	45 965.100	7.342	34 572.447	7.415	6 000
62 804.868	7.434	48 689.215	7.433	36 529.364	7.510	6 500
66 328.011	7.509	51 378.372	7.519	38 456.072	7.599	7 000
69 816.734	7.583	54 036.119	7.601	40 355.636	7.684	7 500
73 274.264	7.653	56 665.414	7.679	42 230.618	7.765	8 000
76 703.304	7.721	59 268.757	7.754	44 083.185	7.841	8 500
80 106.159	7.787	61 848.284	7.826	45 915.194	7.915	9 000
83 484.810	7.851	64 405.841	7.895	47 728.251	7.985	9 500
86 840.982	7.912	66 943.033	7.961	49 523.751	8.052	10 000
90 176.189	7.972	69 461.271	8.025	51 302.924	8.117	10 500
93 491.767	8.030	71 961.799	8.086	53 066.855	8.180	11 000
96 788.909	8.087	74 445.727	8.146	54 816.508	8.240	11 500
100 068.681	8.141	76 914.046	8.204	56 552.748	8.299	11 200

results are listed in *tables III* and *IV*, respectively. The appropriate values of ε are displayed in *table V*. These values have been obtained by Abdullah and Lindsay [23] using data collected from several works on numerous ferromagnetic fluids. The values given are sensitive to both temperature and particle diameter.

TABLE IV
The relation between Chandrasekhar number and solute Rayleigh number for the overstable case for case (b) of solute boundary conditions when both boundaries are rigid. Here $R_t = 1\,000$, $P_m = 4$, $P_r = 1$, $P_r' = 1$.

$\varepsilon = 1$		$\varepsilon = 0.5$		$\varepsilon = 0$		
R_s	a	R_s	a	R_s	a	Q
–	–	–	–	7 683.127	4.982	500
–	–	13 753.430	5.484	11 035.963	5.519	1 000
–	–	17 549.216	5.817	13 913.726	5.873	1 500
26 321.321	6.103	21 051.425	6.076	16 540.119	6.144	2 000
30 555.819	6.278	24 364.563	6.289	19 003.097	6.367	2 500
34 638.241	6.451	27 541.750	6.473	21 348.323	6.507	3 000
38 605.037	6.599	30 614.631	6.636	23 603.207	6.724	3 500
42 479.163	9.734	33 603.951	6.781	25 785.769	6.873	4 000
46 276.492	6.858	36 524.222	6.913	27 908.629	7.007	4 500
50 008.644	6.973	39 386.087	7.035	29 981.044	7.131	5 000
53 684.467	7.080	42 196.636	7.147	32 010.052	7.245	5 500
57 310.904	7.181	44 965.199	7.252	34 001.153	7.351	6 000
60 893.527	7.276	47 693.843	7.350	35 958.749	7.450	6 500
64 436.900	7.365	50 387.702	7.443	37 886.429	7.545	7 000
67 944.815	7.450	53 050.202	7.530	39 737.166	7.632	7 500
71 420.469	7.531	55 684.224	7.613	41 663.457	7.715	8 000
74 866.588	7.609	58 292.216	7.693	43 517.424	7.795	8 500
78 285.518	7.683	60 876.283	7.768	45 350.891	7.871	9 000
81 679.299	7.754	63 438.252	7.841	47 165.437	7.943	9 500
85 049.716	7.822	65 979.716	7.910	48 962.441	8.013	10 000
88 398.345	7.888	68 502.081	7.977	50 743.119	8.080	10 500
91 726.583	7.951	71 006.591	8.041	52 508.546	8.144	11 000
95 035.679	8.013	73 494.357	8.103	54 259.680	8.206	11 500
98 326.750	8.072	75 966.372	8.163	55 997.381	8.266	12 000

TABLE V
Typical values of the strength of the nonlinearity ε.

Ferromagnetic fluid	Particle diameter	ε
Diester (magnetite)	110	1
Water (magnetite)	120	1.18
Petrolum (magnetite)	75	0.38
Kerosene (magnetite)	70	0.31
Fluorocarbon (magnetite)	76	0.4
Diester (iron)	100	4.29
Mercury (iron)	45	0.78

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